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ON THE ERGODICITY OF A CLASS OF SKEW PRODUCTS

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ABSTRACT
Let $\varphi:[0,1]\to\mathbf{R}$ have continuous derivative on the closed interval $[0,1]$, $\int_0^1\varphi(x)dx=0$, and let α be irrational. If $\varphi(1)\neq\varphi(0)$, then $(x,y)\mapsto(x+\alpha,y+\varphi(x))$ is ergodic on $\mathbf{R}/\mathbf{Z}\times\mathbf{R}$.

Introduction

We shall study skew products of the form

$$T_\varphi:\mathbf{R}/\mathbf{Z}\times\mathbf{R}\to\mathbf{R}/\mathbf{Z}\times\mathbf{R},$$

$$T_\varphi(x,y)=(x+\alpha,y+\varphi(x)),$$

where α is an irrational number and $\varphi:\mathbf{R}/\mathbf{Z}\to\mathbf{R}$ is a measurable function. In what follows, we shall have normalized Haar measure on the torus $\mathbf{R}/\mathbf{Z}$, Lebesgue measure on $\mathbf{R}$ and the corresponding product measure λ on the product space $\mathbf{R}/\mathbf{Z}\times\mathbf{R}$.

Suppose we are given a function $\varphi:[0,1]\to\mathbf{R}$, continuous on the closed interval $[0,1]$. If we replace the map $x\mapsto\varphi(x)$ by $x\mapsto\varphi(\{x\})$, $\{x\}$ the fractional part of x, then the latter map will define a skew-product in the above sense. In agreement with this convention we may state the following result:

THEOREM. *Let $\varphi:[0,1]\to\mathbf{R}$ be continuously differentiable on the closed interval $[0,1]$, let $\int_0^1\varphi(x)dx=0$ and suppose that α is irrational.*
If $\varphi(1)\neq\varphi(0)$, then T_φ is ergodic with respect to λ.

COROLLARY. *Suppose that φ and α are as in the theorem. Then the factor S_φ: $\mathbf{R}/\mathbf{Z}\times\mathbf{R}/\mathbf{Z}\to\mathbf{R}/\mathbf{Z}\times\mathbf{R}/\mathbf{Z}$, $S_\varphi(x,y)=(x+\alpha,y+\varphi(x))$ will be ergodic (with respect*

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to normalized Haar measure). In particular, if $\varphi(x) = \beta x - \beta/2$, $\beta \neq 0$ some real number, then S_φ will be ergodic.

This corollary could have been deduced from a well-known result of H. Furstenberg ([3], theorem 2.1), but only for $\beta \neq 0$ an integer.

The proof of our theorem will use heavily results on the uniform distribution modulo one of the sequence $(\{n\alpha\})_{n \geq 0}$ (see the monograph [5]), also some techniques from [1], [2], [3] and [4], but it will not employ a rather "unpleasant" functional equation (see [1], theorem 1; [3], lemma 2.1).

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PROOF OF THE THEOREM. Let φ and α be as in the theorem, $0 < \alpha < 1$. If n is a natural number, then we define

$$\varphi_n(x) = \varphi(x) + \varphi(\{x + \alpha\}) + \cdots + \varphi(\{x + (n-1)\alpha\}) \qquad (x \in \mathbf{R}/\mathbf{Z}).$$

DEFINITION. A number c in $\mathbf{R}$ will be called a *period* of the skew-product T_φ if for every T_φ-invariant function f in $L^\infty(\mathbf{R}/\mathbf{Z} \times \mathbf{R})$ the equality $f(x, y + c) = f(x, y)$ holds λ-almost everywhere.

The set of periods of T_φ is a subgroup of the additive group $\mathbf{R}$. It will be denoted by P_φ. We shall show in our proof that every T_φ-invariant function in $L^\infty(\mathbf{R}/\mathbf{Z} \times \mathbf{R})$ is necessarily constant. This will imply ergodicity of T_φ.

LEMMA 1. *T_φ is ergodic if and only if $P_\varphi = \mathbf{R}$.*

PROOF. The proof of this lemma is straightforward and will be omitted. □

The following lemma shows how to obtain periods.

LEMMA 2. *Let $\varphi : [0,1] \to \mathbf{R}$ be continuously differentiable on the closed interval $[0,1]$, let $\varphi(1) \neq \varphi(0)$, $\int_0^1 \varphi(t)dt = 0$, and let α be irrational, $0 < \alpha < 1$.*

We put

$$\rho(\alpha) = \liminf_{n \to \infty} q_n \cdot |q_n\alpha - p_n|,$$

where p_n/q_n are the n-th order convergents in the regular continued fraction expansion of α. Always $0 \leqq \rho(\alpha) \leqq 1/\sqrt{5}$.

Then for every $c \in I(\alpha, \varphi) :=]-(1-\rho(\alpha))(\varphi(1) - \varphi(0))/2, (1-\rho(\alpha)) \cdot (\varphi(1) - \varphi(0))/2[$ and for almost all x in $\mathbf{R}/\mathbf{Z}$ there exists a subsequence $(q_{n_k})_{k \geq 1}$ of denominators (which will depend on x) such that

$$\lim_{k \to \infty} \sum_{0 \leqq i < q_{n_k}} \varphi(\{x + i\alpha\}) = c.$$

PROOF. We have

$$\alpha = p_n/q_n + \theta_n/a_{n+1}q_n^2,$$

where $\alpha = [0; a_1, a_2, \ldots]$ and $\theta_n = (-1)^n|\theta_n|, |\theta_n| < 1$. It is elementary to see that

$$\varphi_q(x) = \sum_{k=0}^{q-1} \varphi(\{k/q + \bar{x} + m_k\theta/aq^2\}),$$

where $q := q_n$, $a := a_{n+1}$, $\theta := \theta_n$ and $\bar{x} = \{qx\}/q$, $r = [qx]$ and m_k is defined by the congruence $m_k p_n \equiv k - r \mod q_n$, $0 \leqq k \leqq q_n - 1$.

In what follows, n will always be *even*; the case of odd n is completely analogous.

Let $\omega(\delta) = \sup\{|\varphi'(y) - \varphi'(z)| : |y - z| < \delta,\ y \text{ and } z \text{ in } [0,1]\}$, $\delta > 0$, denote the modulus of continuity of φ'. It is easy to prove the following identity:

$$\begin{aligned}\varphi_q(x) = \sum_{k=0}^{q-2} \varphi(k/q) + \sum_{k=0}^{q-2} \varphi'(k/q) m_k\theta/aq^2 + \bar{x}\sum_{k=0}^{q-2}\varphi'(k/q) \\ + \varphi(\{1 - 1/q + m_{q-1}\theta/aq^2 + \bar{x}\}) + O(\omega(2/q)).\end{aligned} \tag{1}$$

We study the different expressions in (1):

PROPOSITION 1. *Let $f:[0,1]\to\mathbf{R}$ be continuously differentiable on the closed interval $[0,1]$ and let N be a natural number. Then*

$$\int_0^1 f(t)dt = (1/N)\sum_{k=0}^{N-1} f(k/N) + (f(1) - f(0))/2N + O(\omega(1/N)/N),$$

ω being the modulus of continuity of f' on $[0,1]$.

The proof of this proposition is elementary.

Let us discuss identity (1) in the special case

$$\rho(\alpha) = \liminf_{n\to\infty} q_n|q_n\alpha - p_n| = \liminf_{n\to\infty}|\theta_n|/a_{n+1} = 0.$$

Suppose we can choose a subsequence $(n_k)_{k\geqq 1}$ such that

$$\lim_{k\to\infty}|\theta_{n_k}|/a_{n_k+1} = 0, \qquad \text{all } n_k \text{ are even.}$$

Then identity (1) holds and we see: the first sum is equal to

$$-\varphi(1 - 1/q) - (\varphi(1) - \varphi(0))/2 + O(\omega(1/q)).$$

The second sum will become arbitrarily small for $k \to \infty$. The third sum is equal to

$$\{qx\} \cdot (\varphi(1) - \varphi(0)) + O(\omega(1/q)).$$

The number $\{1 - 1/q + m_{q-1}\theta/aq^2 + \bar{x}\}$ will tend to one.

The sequence $(\{q_{n_k}x\})_{k \geqq 1}$ is uniformly distributed modulo 1 for almost all x in $\mathbf{R}/\mathbf{Z}$. Thus, for every $d \in [0,1[$, and almost all x, there exists a subsequence (which will again be denoted by $(q_{n_k})_{k \geqq 1}$; it depends on x) with $\lim_{k \to \infty}\{q_{n_k}x\} = d$. As a consequence,

$$\lim_{k \to \infty} \varphi_{q_{n_k}}(x) = (\varphi(1) - \varphi(0))(2d - 1)/2.$$

If a subsequence $(n_k)_{k \geqq 1}$ of the above type with n_k even for all k does not exist, then we calculate identity (1) for odd n and continue as indicated.

Suppose from now on that $\rho(\alpha) > 0$. We study the second sum in (1):

Proposition 2. *Let $g : [0,1] \to \mathbf{R}$ be a continuous function on $[0,1]$ with modulus of continuity ω and let s and q, $s < q$, be two natural numbers with continued fraction expansion $s/q = [0; b_1, \ldots, b_N]$. If $b_i \leqq A$, $1 \leqq i \leqq N$, then*

$$\left| (1/q) \sum_{k=1}^{q} g(k/q)\{ks/q\} - \tfrac{1}{2}\int_0^1 g(t)dt \right| \leqq C(g) \cdot \omega([(A \log q/q)^{-1}]^{-1/2})$$

where $C(g)$ is a constant depending only on g.

To prove this proposition, one uses well-known estimates for the integration error for the continuous function $F(x,y) = g(x)y$ on $[0,1]^2$ and discrepancy estimates for the finite sequence $(k/q, \{ks/q\})_{k=1}^{q}$ (see [5], chapter 2, §5).

Proposition 3. *Let $(q_{n_k})_{k \geqq 1}$ be a subsequence of $(q_n)_{n \geqq 1}$ such that*

$$\lim_{k \to \infty} |\theta_{n_k}|/a_{n_k+1} = \rho(\alpha)$$

and all n_k are even. Then

$$\lim_{k \to \infty} (\theta_{n_k}/a_{n_k+1}q_{n_k}) \sum_{0 \leqq j \leqq q_{n_k}-2} \varphi'(j/q_{n_k})m_j/q_{n_k} = (\varphi(1) - \varphi(0))\rho(\alpha)/2.$$

If $\rho(\alpha) > 0$ then there exists a constant A such that $a_i \leqq A$ for all i. If n is even, then one can write the sequence $(m_j/q_n)_{j=0}^{q_n-1}$ in the form $(\{(j-r)s_n/q_n\})_{j=0}^{q_n-1}$ with $s_n = q_n - q_{n-1}$. Hence there exists a constant B such that for every n and every continued fraction coefficient b_i of s_n/q_n we have $b_i \leqq B$. We finally apply

Proposition 2 (the slight perturbation of the sequence $(\{js/q_n\})_{j=0}^{q_n-1}$ does not change the order of the estimate in Proposition 2).

Let us now complete the proof of Lemma 2. Suppose that we can find a subsequence $(q_{n_k})_{k\geqq 1}$ of $(q_n)_{n\geqq 1}$ such that

$$\lim_{k\to\infty} |\theta_{n_k}|/a_{n_k+1} = \rho(\alpha), \qquad n_k \text{ even for all } k. \tag{2}$$

Let d be an arbitrary number in the interval $[0, 1-\rho(\alpha)[$. Due to the uniform distribution in $\mathbf{R}/\mathbf{Z}$ of the sequence $(\{q_{n_k}x\})_{k\geqq 1}$ for almost all x in $\mathbf{R}/\mathbf{Z}$ there exists a subsequence, which will again be denoted by $(q_{n_k})_{k\geqq 1}$, such that

$$\lim_{k\to\infty} \{q_{n_k}x\} = d.$$

This subsequence clearly depends on x. We now put $q = q_{n_k}$ in (1) and let $k\to\infty$. This gives

$$\lim_{k\to\infty} \varphi_{q_{n_k}}(x) = (\varphi(1)-\varphi(0))(-1+\rho(\alpha)+2d)/2.$$

If in (2) there are only finitely many even indices n_k, then one calculates identity (1) for odd n and proves the equivalent of Proposition 3 for odd n_k (in this case one chooses $s = q_{n-1}$ in the proof). The proof of Lemma 2 is then completed in the very same manner as above. □

LEMMA 3. $I(\alpha,\varphi)\subset P_\varphi$.

PROOF. Let f be an arbitrary T_φ-invariant function in $L^\infty(\mathbf{R}/\mathbf{Z}\times\mathbf{R})$ and let $c\in I(\alpha,\varphi)$. Then, λ-almost everywhere,

$$f(x+n\alpha, y+c) = f(x, y+c-\varphi_n(x)) \tag{3}$$

for all $n\in\mathbf{N}$. According to Lemma 2 there exists a sequence of measurable functions $(\varepsilon_k)_{k\geqq 1}$, $\varepsilon_k:\mathbf{R}/\mathbf{Z}\to\mathbf{N}$, such that

$$\lim_{k\to\infty} q_{\varepsilon_k(x)}\cdot\alpha = 0 \quad (\text{mod } 1),$$

$$\lim_{k\to\infty} \varphi_{q_{\varepsilon_k(x)}}(x) = c \quad \text{almost everywhere},$$

namely $\varepsilon_k(x) := \min\{n : |c-\varphi_{q_n}(x)| < 1/k\}$, $k = 1,2,\ldots$. We now replace in (3) n by $\varepsilon_k(x)$ and let k tend to infinity. On the left side we consider the limit in

$L^1(\mathbf{R}/\mathbf{Z})$ (y fixed), on the right side in $L^1([-N,N])$ (x fixed, $N \in \mathbf{N}$ arbitrary). We finally get

$$f(x, y + c) = f(x, y) \qquad \lambda\text{-almost everywhere.} \qquad \square$$

Lemma 3 clearly implies $P_\varphi = \mathbf{R}$, which ends the proof of the theorem.

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